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Author's addresses Michael Breuer
E-mail: mbreuer@soi.unizh.ch

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Rämistrasse 71
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Phone: +41-1-634 21 37
Fax: +41-1-634 49 82
URL: www.soi.unizh.ch
E-mail: soilib@soi.unizh.ch

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Michael Breuer*
University of Zurich

ABSTRACT

In the literature on optimal indemnity schedules, indemnities are usually restricted to be non-negative. Gollier (1987) shows that this constraint might well bind: insured could get higher expected utility if insurance contracts would allow payments from the insured to the insurer at some losses. However, due to the insurers' cost function Gollier supposes, the optimal insurance contract he derives underestimates the relevance of the non-negativity constraint on indemnities. This paper extends Gollier's findings by allowing for negative indemnity payments for a broader class of insurers' cost functions.

Keywords: Insurance, Indemnity, Deductible, Co-Insurance

JEL-Classification: D 80, D81, D 89

* Address for correspondence: Socioeconomic Institute, University of Zurich, Hottingerstr. 10, 8032 Zurich, Switzerland. E-mail: mbreuer@soi.unizh.ch. Phone: +41 1 634 45 95.
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1. Introduction

In insurance economics there is a vast literature on optimal indemnity schedules.[1] A common exercise in this literature is to restrict feasible indemnity schedules in several ways. First, premiums are supposed to recover at least expected indemnity payments. This restriction is sensible since it guarantees non-negative profits of insurers. It can be interpreted as the participation constraint of the insurer. Second, indemnity payments must not exceed losses, or (with some loss of generality) marginal indemnities must be smaller than 1. While this restriction is most relevant in practice, where insurers have to be concerned about moral hazard, it is not clear why it is also imposed in simpler models that abstract from moral hazard.[2] Third, and finally, indemnities have to be non-negative. This assumption looks most sensible, since it prevents risk averse insured to become insurers themselves. However, as Gollier (1987) points out, this restriction can be binding for some loss distributions. In other words, under certain circumstances, the insured can get higher expected utility if they are allowed to sign contracts that provide for payments from the insured to the (risk neutral!) insurer for some losses.

Specifically, Gollier obtains the following results for insurance contracts that do not impose a non-negativity constraint on indemnities:

1. As in insurance contracts of the usual deductible type (Arrow, 1971), optimal contracts show a (non-negative) loss x_+ that acts as a deductible. For all losses above the deductible marginal indemnity is 1 and indemnity amounts to $I(x) = x - x_+$ for $x > x_+$.

2. Optimal contracts might contain a (non-negative) loss x_- with $x_- \leq x_+$. For all losses between zero and x_- indemnity is negative and marginal indemnity equals 1. Consequently, indemnity payments are given by $I(x) = x - x_-$ for $x < x_-$.

[1] See for example Mossin (1968); Gould (1969); Arrow (1971); Moffet (1977); Raviv (1979); Drèze (1981); Schlesinger (1981); Gollier and Schlesinger (1996); Spaeter and Roger (1997). Many more contributions deal with the consequences of asymmetric information on optimal indemnity schedules.

3. For all losses between x_- and x_+ indemnities are zero

4. For the lower bound it is true that $x_- \leq F(1/2)$, with $F(x)$ representing the cumulative distribution function of losses x. The practical consequence of this result is that the non-negativity constraint is never binding if the probability of suffering a loss is less than $1/2$, which is obviously the case for many insured incidents.

The aim of this paper is to show that Gollier's results partly depend on the special form of a restriction he imposes on the insurer's cost function and that the non-negativity constraint is more likely to bind if we allow for more general cost functions. Gollier assumes costs C depending on the expected value of absolute indemnities $|I(x)|$ transferred between insurer and insured $\left(C\left(E(|I(x)|)\right)\right)$. This cost structure reflects the assumption that the insurer has to bear fixed costs only (e.g. for hiring staff and renting offices before knowing the actual value of the indemnities). However, it is more plausible to assume that costs also depend on indemnities actually transferred. This calls for a more flexible cost function which will turn out to change some of Gollier's results considerably.

2. The model

While Gollier gets his results by applying calculus of variation, this paper will (in line with Raviv (1979)) employ optimal control theory.

2.1 Assumptions

Let risk averse individuals have utility function $U(A)$, $U'(A) > 0$, $U''(A) < 0$, with A representing their net wealth. The risk neutral insurer is supposed to recover cost but to make zero expected profit. Premiums (P) therefore are equal to indemnities paid plus administrative costs that also emerge when indemnities are negative:

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2 See Huberman, Mayers and Smith (1983), who derive an optimal indemnity schedule for a concave cost function containing a vanishing deductible and a marginal indemnity greater than 1.

(1) $$P=\int_0^L \left(I(x)+C\left(\left|I(x)\right|\right)\right)f(x)dx,$$

with $f(x)$ representing the density function of losses. We impose a maximal loss of L. In order to allow for increasing marginal costs $C'\left(\left|I(x)\right|\right)\geq 0$ and $C''\left(\left|I(x)\right|\right)\geq 0$.

In contrast, Gollier assumes costs to amount to $C\left(E(\left|I(x)\right|)\right)$. Consequently, in his model the premium reads as

(2) $$P_{Gol}=\int_0^L I(x)f(x)dx+C\left(\int_0^L \left|I(x)\right|f(x)dx\right).$$

Observe that (1) is compatible with Gollier's premium function (2) if marginal costs are constant $(C''=0)$. Therefore (1) is indeed a generalization of (2).

Let w denote individuals' exogenous wealth. Insured's expected utility

(3) $$\int_0^L U\left(w-P-x+I(x)\right)f(x)dx$$

is to maximize subject to (1). The constraint $I(x)\leq x$ is disregarded for two reasons: First, as pointed out before, this constraint does not make much sense in a model which does not allow for informational asymmetries, specifically moral hazard. Second, it will turn out that this restriction is not binding anyway if costs are convex.

2.2 The optimal indemnity schedule

In order to solve this problem using optimal control, we introduce the following state variable:[3]

(4) $$\Gamma(\hat{x})=-\int_0^{\hat{x}} \left(I(x)+C\left(\left|I(x)\right|\right)\right)f(x)dx.$$

[3] For mathematical reference see Chiang (1992, chapter 10).

The initial condition is that $\Gamma(0) = 0$. The terminal condition reads as $\Gamma(L) = -P$, which corresponds to a zero-profit constraint for the insurer. The corresponding Hamiltonian reads as

$$(5) \qquad H = U(w - P - x + I(x))f(x) - \lambda(x)(I(x) + C(|I(x)|))f(x).$$

Since the Hamiltonian does not depend on the state variable $\lambda'(x) = -\frac{\partial H}{\partial \Gamma} = 0$, i.e. λ is a constant. To find the optimal indemnity schedule, the Hamiltonian is differentiated w.r.t. $I(x)$. After rearranging terms, one has:

$$(6) \qquad U'(w - P - x + I(x)) = \lambda(1 + C'(I(x)) \cdot sign(I(x))).$$

For negative (positive) indemnities, this can be simplified to

$$(7) \qquad U'(w - P - x + I(x)) = \lambda(1 - C'(-I(x))) \qquad \text{and}$$

$$(8) \qquad U'(w - P - x + I(x)) = \lambda(1 + C'(I(x))),$$

respectively. Eliminating λ and combining (7) and (8) yields

$$(9) \qquad \left.\frac{U'(w - P - x + I(x))}{(1 - C'(-I(x)))}\right|_{I(x)<0} = \left.\frac{U'(w - P - x + I(x))}{(1 + C'(I(x)))}\right|_{I(x)>0}.$$

As can be seen from (9), negative indemnities are restricted: The marginal costs they induce must be lower than 1. At $I(x) = 0$, (9) might be rewritten as

$$(10) \qquad \frac{U'(w - P - x_-)}{(1 - C'(0))} = \frac{U'(w - P - x_+)}{(1 + C'(0))}.$$

According to (10) $x_- = x_+$ for $C'(0) = 0$. However, for positive marginal costs $(C'(0) > 0)$, the denominator on the rhs of (10) is greater than the denominator on the lhs. To compensate for this difference, $U'(w - P - x_+)$ must be greater than $U'(w - P - x_-)$. Under the assumption of decreasing marginal utility, this can only be the case if $x_- < x_+$. As negative indemnities are restricted, no positive lower bound x_- can be determined if $C'(-I(0)) \geq 1$. Therefore, in this paper it is assumed that marginal costs $C'(-I(0))$ are strictly smaller than 1.

From (9) and (10) follows:

- The optimal indemnity schedule contains a lower bound x_- and an upper bound (i.e. a deductible) x_+. For losses lower than x_- indemnities are negative (the insured pays the insurer); for losses exceeding x_+ indemnities are positive.
- For losses between x_- and x_+ no transfer between insurer and insured takes place.
- The distance between x_- and x_+ depends on marginal costs at $I(x)=0$ and the insured's risk aversion. The more risk averse insured, the smaller the range of losses they have to bear completely.

For increasing marginal costs full marginal indemnity is not generally optimal. Instead, $\frac{\partial I(x)}{\partial x} \le 1$. This can be shown by differentiating (6) w.r.t. x:

$$\begin{aligned} & U''(w-P-x+I(x)) \cdot \left(-1+\frac{\partial I(x)}{\partial x}\right) = \\ & \lambda\left(C''(|I(x)|) \cdot \frac{\partial I(x)}{\partial x}\right) \end{aligned} . \tag{11}$$

Solving for $\frac{\partial I(x)}{\partial x}$ and substituting for λ from (6) gives the marginal indemnity

$$\frac{\partial I(x)}{\partial x} = \frac{\left(1+C'(|I(x)|) \cdot sign(I(x))\right) \cdot U''(A)}{\left(1+C'(|I(x)|) \cdot sign(I(x))\right) \cdot U''(A) - U'(A) \cdot C''(|I(x)|)}, \tag{12}$$

with $A = w-P-x+I(x)$.

Finally, using the definition of absolute risk aversion $Ra(A) = -\frac{U''(A)}{U'(A)}$ results in:

$$\frac{\partial I(x)}{\partial x} = \frac{Ra(A)}{Ra(A) + \frac{C''(|I(x)|)}{1+C'(|I(x)|) \cdot sign(I(x))}} . \tag{13}$$

Remember from (9) that $C'(-I(x)) \le 1$. Therefore, $0 \le \frac{\partial I(x)}{\partial x} \le 1$. Specifically:

- Marginal indemnity increases with insured's risk aversion. As risk aversion approaches infinity, full marginal reimbursement becomes optimal.

- Constant marginal costs turn out to be special case of (13) with $C''=0$, giving rise to full marginal reimbursement.

However, in general optimal indemnity schedules will call for less than full marginal indemnity. Instead, the optimal indemnity schedule will look somewhat like shown in figure 1:

Figure 1: Optimal indemnity with increasing marginal costs

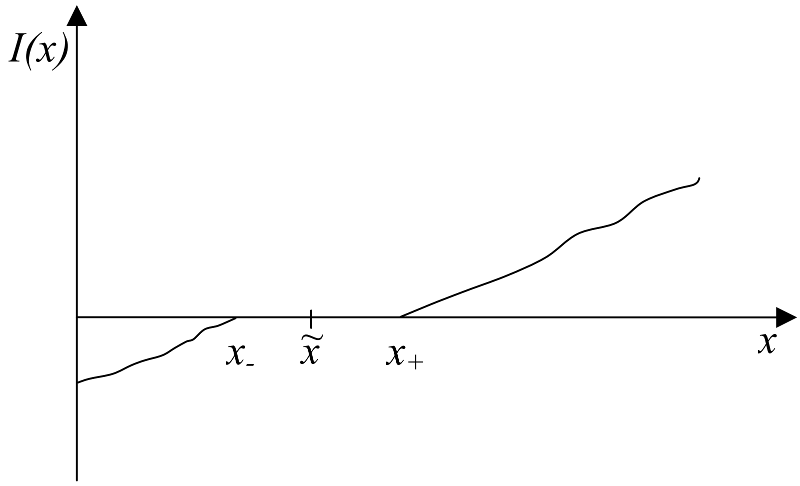

Figure 1 illustrates the results obtained so far. From (10) it is known that x_- and x_+ coincide for infinitely risk adverse individuals. In figure 1 this point is labelled $\hat{x}$. It will be determined in more detail in the next section. However, for not infinitely risk averse individuals the two limits x_- and x_+ are on the left hand side and the right hand side of $\tilde{x}$, respectively. For losses $x < x_-$ indemnity is paid from the insured to the insurer (indemnities are negative). For losses $x > x_+$ indemnities are paid from the insurer to the insured (indemnities are positive). Losses between x_- and x_+ are borne by the insured alone. Marginal indemnity is $0 \le \partial I(x)/\partial x \le 1$ for all losses: For losses lower than x_- insured can partly reduce their payments to the insurer but not for the full amount of the loss; a marginal increase of the loss will only entitle insured to reduce their payments by less than the marginal increase of the loss. For losses greater than x_+ the insured are entitled to receive positive indemnity payments from the insurer. However, marginal indemnity will again be lower than 1 so that insured still have to bear a marginal loss partly.

3. More detailed characterization of the optimal contract

Having derived the main properties of an insurance contract without the non-negativity constraint on indemnities, it is useful to further explore the terms of the optimal insurance contract. In particular, since $x_- \leq x_+$ the upper bound for x_- and the lower bound for x_+ are of interest. Since x_- and x_+ coincide for infinitely risk adverse insured, both bounds have the same value labelled $\hat{x}$ in figure 1.

To determine $\tilde{x}$ remember that premiums depend on the actuarially fair value of expected (net-)indemnity payments plus administrative cost, which rise as transfers between insurer and insured rise. The former do not change insured's expected wealth. In contrast, insured's losses due to administrative costs are lower if transactions between insurer and insured are reduced. While individuals' risk aversion determines the distance between x_- and x_+ as well as the slope of the indemnity function, the critical value $\tilde{x}$ depends on the administrative cost function alone. Expected transaction costs are minimized for any indemnity schedule by a loss $\tilde{x}$ that minimizes

$$(14) \qquad \int_0^L C\left(\left|I(x-\tilde{x})\right|\right) f(x)dx \, .$$

If no transfer between insurer and insured takes place, $C(|I(0)|)=0$. Differentiating (14) w.r.t. $\tilde{x}$ yields the necessary condition

$$(15) \qquad \int_0^L -C'\left(\left|I(x-\tilde{x})\right|\right) \cdot I'(x-\tilde{x}) \cdot sign(x-\tilde{x}) f(x)dx = 0 \, ,$$

which is more readable if written as

(16) $$\int_0^{\tilde{x}} C'\left(\left|I(x-\tilde{x})\right|\right)\cdot I'(x-\tilde{x})f(x)dx - \int_{\tilde{x}}^{L} C'\left(\left|I(x-\tilde{x})\right|\right)\cdot I'(x-\tilde{x})f(x)dx = 0 .$$

If marginal costs are constant, marginal indemnity equals 1 (see equation (13)) and $\tilde{x}$ always coincides with the median of the loss distribution.[4] Consequently, in this case the non-negativity constraint is never binding if the probability of loss is lower than $1/2$.

However, according to (16) in general $\tilde{x}$ will deviate from the median. For asymmetric distributions with more mass on low losses, $\tilde{x}$ will be on the right hand side of the median for the following reasons:

- High losses can deviate more from $\tilde{x}$ than low losses, causing higher absolute indemnities than low losses do.
- Higher indemnities go along with higher marginal costs.
- From (9), marginal indemnities for low losses are restricted.

All this will cause the weight of the second term in (16) to get higher than it would be under constant marginal indemnities and constant marginal costs. To balance out both terms, $\tilde{x}$ has to move to the right of the median. Consequently, for convex costs it is not true in general that the non-negativity constraint is not binding if the probability of loss is lower than $1/2$. For sharply increasing marginal costs, $\tilde{x}$ may deviate from the median considerably. An imposed non-negativity constraint therefore will be binding more often than Gollier suggests. The effect of a non-negativity constraint to the insured is that they are urged to accept higher marginal costs to reduce the variance of their final wealth. This effect becomes most obvious for insured with risk aversion approaching infinity, inducing full marginal indemnity[5]. In order to stabilize their final income they can

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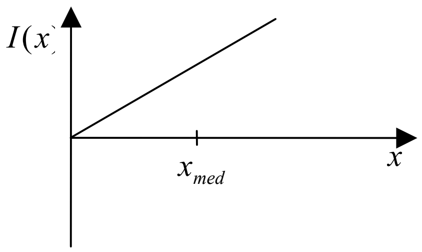
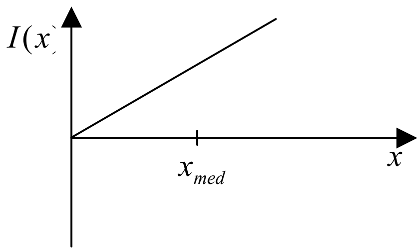
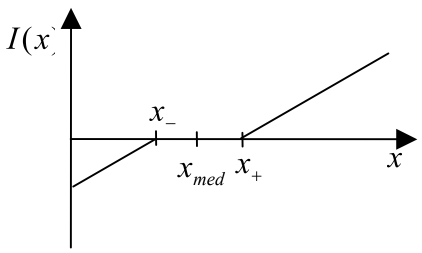
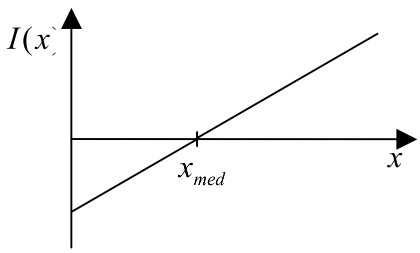
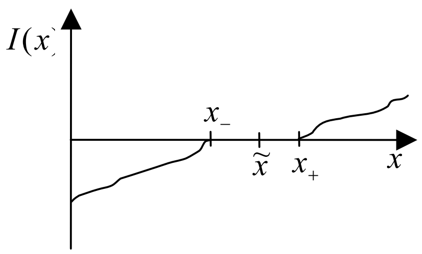
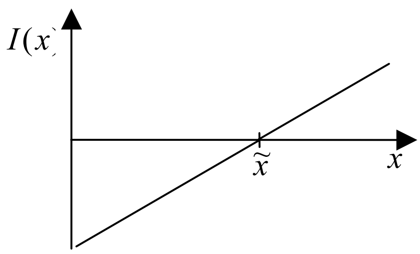
[4] If marginal indemnity is 1 and $x_- = x_+ = \tilde{x}$, the term $C'\left(\left|I(x-\tilde{x})\right|\right)$ simplifies to $C'\left(\left|x-\tilde{x}\right|\right)$ This allows to rewrite (16) as $C'\left(\left|x-\tilde{x}\right|\right)\cdot\left(\int_x^{\tilde{x}} f(x)dx - \int_{\tilde{x}}^{L} f(x)dx\right) = C'\left(\left|x-\tilde{x}\right|\right)\cdot\left(2F(\tilde{x})-1\right) = 0$, which is zero if $\tilde{x}$ takes the value of the median or if marginal costs are zero.

[5] This result is due to (13). See also table 1.

only reduce their final losses to zero and have to bear the high marginal costs of the high indemnity payments from the insurer.

Table 1 summarizes the range of optimal indemnity schedules for different degrees of risk aversion and different cost functions by highlighting some extreme cases for an asymmetric loss distribution with more mass on low losses. If marginal costs are zero (cases (a) and (b)) the insured will insure their full wealth or buy no insurance at all (if confronted with high fixed costs of the insurance contract, e.g. provision for the agent). If marginal costs are positive but constant (cases (c) and (d)), the insured will always opt for full marginal indemnity. As risk aversion approaches infinity, x_- and x_+ will tend towards the median of the loss distribution as in case (d). Case (e) represents the standard indemnity schedule for non-constant but finite marginal costs and finite risk aversion. Note that $\tilde{x}$ is right off the median. Consequently, the optimal indemnity schedule for infinitely risk averse individuals cuts the abscissa at $\tilde{x}$ rather than at x_{med} (case (f)).

Table 1: Optimal indemnity schedules without the non-negativity constraint on indemnities

	$1 < Ra(A) < \infty$	$Ra(A) \to \infty$
$C' = 0$	(a) 	(b) 
$C' > 0$ $C'' = 0$	(c) 	(d) 
$C' > 0$ $C'' > 0$	(e) 	(f) 

4. Conclusion

The non-negativity constraint on indemnities is common in insurance economics. Gollier (1987) was the first to show that this constraint may be binding under a certain cost function. By deriving the properties of an optimal insurance contract for a broader class of convex cost functions, this paper shows that relaxing the non-negativity constraint affects the optimal insurance contract in ways that have not been recognized before. It has been shown that optimal marginal indemnity will be smaller or equal 1. Negative indemnities might be restricted. Furthermore, the optimal contract might contain negative indemnity payments even if probability of loss is less than $1/2$.

These results prove that our analysis is more than an academic exercise. They make insurance contracts allowing for negative indemnities interesting for a broader class of insured incidents. For these incidents, individuals could get insurance coverage by accepting some loss around a critical value at any state of the world at considerably lower costs than a contract would cause that does not allow for negative indemnities.

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